NP-Hardness, Part 3

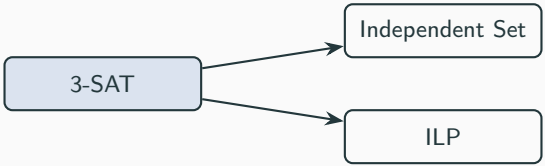
Clique, Vertex Cover

Recap

To show a problem B is NP-complete:

1. $B \in$ NP: a certificate and a polynomial-time verifier;
2. a reduction $A \leq_p B$ from a known NP-complete A.

So far:



Today: two problems closely related to Independent Set. Each reduction is a one-line transformation of the graph.

Clique

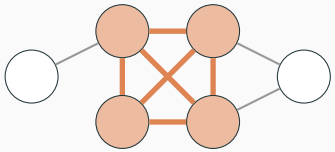
Clique

A clique in $G = (V, E)$ is a set $S \subseteq V$ in which *every* pair of vertices is joined by an edge.

Clique

Given G and k, does G have a clique of size $\geq k$?

In NP. Certificate: the set S. Check $|S| \geq k$ and all $\binom{|S|}{2}$ pairs, $O(n^2)$ time with an adjacency matrix.

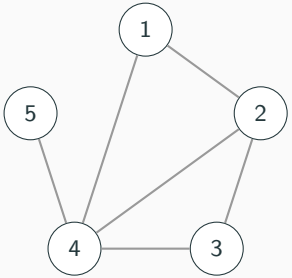


A clique of size 4.

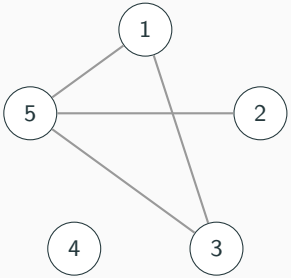
Independent set: no pair adjacent. Clique: every pair adjacent. Opposites!

The Complement Graph

The complement $\bar{G}$ has the same vertices, and an edge exactly where G has none. Computing it takes $O(n^2)$ time.



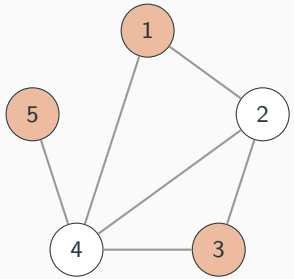
G



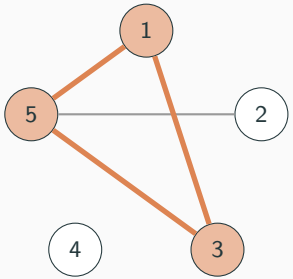
$\bar{G}$

The Complement Graph

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G



$\bar{G}$

$\{1, 3, 5\}$ is an independent set in G and a clique in $\bar{G}$.

Independent Set $\leq_p$ Clique

Observation

S is an independent set in $G \iff S$ is a clique in $\bar{G}$.

Proof. For distinct $u, v \in S$: $\{u, v\} \notin E \iff \{u, v\} \in \bar{E}$. So "no pair is an edge of G" is the same as "every pair is an edge of $\bar{G}$". □

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Theorem

Clique is NP-complete.

Proof. Clique $\in$ NP (above). Reduction: map an Independent Set instance (G, k) to the Clique instance $(\bar{G}, k)$, in $O(n^2)$ time. By the observation, G has an independent set of size $\geq k \iff \bar{G}$ has a clique of size $\geq k$. Independent Set is NP-complete, so Clique is NP-hard. □

Vertex Cover

Vertex Cover

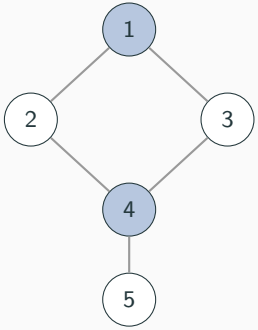
A vertex cover of $G = (V, E)$ is a set $C \subseteq V$ that touches every edge: for every $\{u, v\} \in E$, $u \in C$ or $v \in C$.

Vertex Cover

Given G and k, does G have a vertex cover of size $\leq k$?

In NP. Certificate: the set C. Check $|C| \leq k$ and scan every edge, $O(n + m)$ time.

Think: place guards at vertices so every road (edge) is watched.



$\{1, 4\}$ (blue) covers all 5 edges.

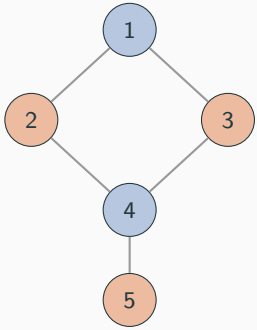
Covers and Independent Sets Are Complements

Observation

S is an independent set $\iff$ $V \setminus S$ is a vertex cover.

($\Rightarrow$) Take any edge $\{u, v\}$. Not both ends are in S, so at least one is in $V \setminus S$.

($\Leftarrow$) If some edge had both ends in S, neither end would be in $V \setminus S$, which would then miss that edge. $\square$



cover $\{1, 4\}$ (blue),
independent set $\{2, 3, 5\}$ (orange)

Independent Set $\leq_p$ Vertex Cover

Theorem

Vertex Cover is NP-complete.

Proof. Vertex Cover $\in$ NP (above). Reduction: map an Independent Set instance (G, k) to the Vertex Cover instance $(G, n - k)$: the same graph, with target $n - k$.

By the observation,

$$G \text{ has an independent set of size } \geq k \iff G \text{ has a vertex cover of size } \leq n - k.$$

Independent Set is NP-complete, so Vertex Cover is NP-hard. $\square$

Independent Set $\leq_p$ Vertex Cover

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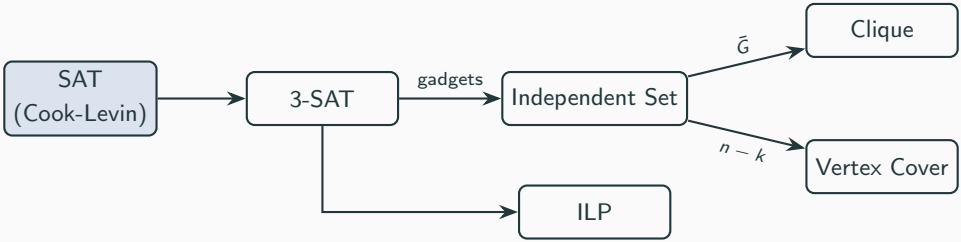
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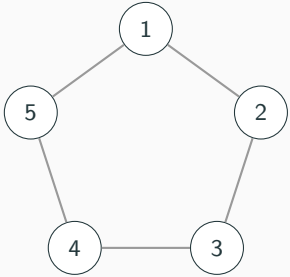
Note. *Minimum* vertex cover and *maximum* independent set are the same problem in disguise: min cover $= n -$ max independent set.

The Reductions So Far



- Every problem here is NP-complete, so each reduces to every other one (through Cook-Levin, if not directly).
- Many more are known: Hamiltonian Cycle, 3-Coloring, Subset Sum, Traveling Salesman, ... (Karp's 21 problems, 1972).
- The more problems we know are NP-complete, the easier it is to find one close to a new problem to reduce from.

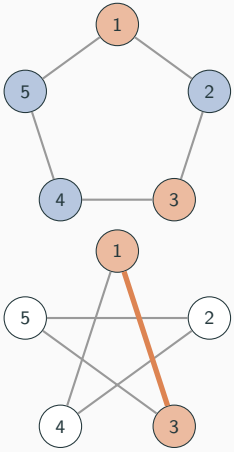
Exercise



The 5-cycle C_5

1. For C_5, find a maximum independent set, a minimum vertex cover, and a maximum clique of $\bar{C}_5$. Check the relations from today.
2. Give a direct reduction Vertex Cover $\leq_p$ Clique.
3. A friend shows Vertex Cover $\leq_p$ ILP. Is that a new proof that ILP is NP-hard?
4. A friend shows ILP $\leq_p$ Vertex Cover. Does that show ILP is NP-hard?

Solution



C_5 and $\bar{C}_5$

(a) Max independent set $\{1, 3\}$, size 2 (three of five vertices on a 5-cycle always include two neighbors). Min cover $\{2, 4, 5\}$, size $3 = 5 - 2$. $\bar{C}_5$ is again a 5-cycle; its max clique $\{1, 3\}$ has size 2, the same as the max independent set of C_5.

(b) $(G, k) \mapsto (\bar{G}, n - k)$: G has a cover of size $\leq k \iff$ an independent set of size $\geq n - k \iff \bar{G}$ has a clique of size $\geq n - k$.

(c) Yes. Vertex Cover is NP-complete, so reducing it to ILP shows ILP is NP-hard (another route besides 3-SAT).

(d) No, wrong direction. It shows ILP is no harder than Vertex Cover, which says nothing about ILP being hard.

Summary

- **Clique**: every pair adjacent. Independent sets of G are exactly the cliques of $\bar{G}$, so $(G, k) \mapsto (\bar{G}, k)$.
- **Vertex Cover**: touch every edge. S is independent $\iff$ $V \setminus S$ is a cover, so $(G, k) \mapsto (G, n - k)$.
- Both are in NP, so both are NP-complete.
- To prove a new problem hard, reduce from a known NP-complete problem that looks similar, and check both directions of the "if and only if".